**Distance Magic Labeling of Cycles**

**Chaithra K**

 **Assistant Professor**

 **Dept. of Mathematics,**

 **NMAM Institute of Technology, Nitte-574110.**

**(Affiliated to VTU , Belgavi)**

**Karnataka, India**

**Email:89chaithrak@nitte.edu.in**

**Shankaran P**

**Professor**

**Dept. of Mathematics,**

**NMAM Institute of Technology, Nitte-574110**

**Affiliated to VTU , Belgavi)**

**Karnataka, India**

**Email:shankar@nitte.edu.in**

**Abstract**

Let G =G (V, E) be a graph. If for each vertex v, sum of the labeling of the vertices which are at a distance D from v is constant, then such a labeling is said to be D-distance magic labeling and a graph G is said to be D-distance magic graph. In this paper, we study D-distance magic labeling of cycles, join of cycles and complete graphs and composition of graphs.

**AMS Subject Classification**: 05C78.

**Keywords:** *distance magic**labeling, magic constant, complete graph., regular graph.*

I. Introduction

The concept of distance magic labeling of a graph has been motivated by the construction of magic squares. Because of the historical interest in magic squares, in 1963, Sedlacek introduced magic labeling of graph G= G (V, E).

**Definition 1.1** [5] : A bijection f from the edge set E to a set of positive integers such that

$f\left(e\_{i}\right)\ne f\left(e\_{j}\right) $for all distinct $e\_{i},e\_{j}\in E $and $\sum\_{e\in N\_{E}\left(x\right)}^{}f\left(e\right)$ is same for every $x\in V$, where $N\_{E}\left(x\right)$ is the set of edges incident to X.

In 1994, Vilfred and in 2003 Miller et.al separately introduced distance magic labeling.

**Definition 1.2** [8]: A distance magic labeling is a bijection $f:V\rightarrow \{1,2,…v\}$ with the property that there is a constant k such that at any vertex x,$ \sum\_{y\in N\left(x\right)}^{}f\left(y\right)=k$, where N(x) is the open neighborhood of x.

Later Jinah introduced variations of distance magic labeling. Here instead of open neighborhood he took closed neighborhood. The concept was independently studied by Simanjuntak, Rodgers and Miller. In particular properties of D-distance magic labeling for a distance set D.

**Definition 1.3** [8] : A bijection $f:V\rightarrow \left\{1,2,…,v\right\}$ is said to be a D-distance magic labeling (D-DML) if there exists a constant k such that for any vertex x,$ w\left(x\right)=\sum\_{y\in N\_{D}\left(x\right)}^{}f\left(y\right)=k where N\_{D}\left(x\right)=\{y\in {V}/{d}\left(x,y\right)\in D\}$. A graph which admits D-DML is called D-distance magic graph (D-DMG).

**Definition 1.4** [6] : Let G and H be two graphs where {$x\_{1},x\_{2},….,x\_{p}$} are vertices of G. Based upon the graph G, an isomorphic copy $H\_{j}$ of H replaces every vertex $x\_{j}$ for j=1, 2,…,p, in such a way that each vertex in $H\_{j}$ is joined to all vertices corresponding to the neighbors of the original vertex $x\_{j}$ of G. Let G [H] denote the resulting and called as composition.

**Definition 1.5** [3] :The join of two graphs G and H having disjoint point set $V\_{1}$ and $V\_{2}$ respectively is denote G+H and consists of G$∪$H and all lines joining $V\_{1}$ with $V\_{2}$

For various graph theoretical notations and terminology we refer to F.Harary [3] and D.B.West[2].

II. Main Results

**Theorem 2.1**: A cycle $C\_{4}$ or a disjoint union of $C\_{4}$ is (0, 2)-distance magic graph.

**Proof** : A cycle $C\_{4}$ is (0, 2) –distance graph (see fig:1)

1

3

4

2

 Figure 1

Let G be a disjoint union of k number of $C\_{4}'$s with n vertices.

Then n=4k

Let $x\_{i}$,$ y\_{i}$,$ u\_{i}$,$ v\_{i}$ are the vertices of $i^{th}C\_{4}$.

Let us assume $u\_{i}$,$ v\_{i}$ are at a distance one from $x\_{i}$ , then

$N\_{0,2}\left(x\_{i}\right)=N\_{0, 2}\left(y\_{i}\right)=\left\{x\_{i},y\_{i}\right\}, N\_{0,2}\left(u\_{i}\right)=N\_{0,2}\left(v\_{i}\right)=\left\{u\_{i},v\_{i}\right\}$, i= 1, 2,.…, k.

Now label the vertices as follows.

$$ l\left(x\_{1}\right)=n, l\left(y\_{1}\right)=1$$

$$l\left(u\_{1}\right)=n-1, l\left(v\_{1}\right)=2$$

$$l\left(x\_{2}\right)=n-2, l\left(y\_{2}\right)=3$$

.

.

.

$$l\left(x\_{k}\right)=2k+2, l\left(y\_{k}\right)=2k-1$$

$$l\left(u\_{k}\right)=2k+1, l\left(v\_{k}\right)=2k$$

Observe that $w\left(x\_{i}\right)=w\left(y\_{i}\right)=w\left(u\_{i}\right)=w\left(v\_{i}\right)=n+1, i=1, 2…,k.$

**Theorem 2.2**: For even n & r, a r-regular graph is not (0,1)-distance magic graph.

**Proof:** If G is r-regular (0, 1)-graph then $K^{'}=\frac{\left(r+1\right)\left(n+1\right)}{2}$ [6] where $K^{'}$ is a magic constant.

When n& r both are even (r+1)(n+1) is odd.

Then $K^{'}=\frac{\left(r+1\right)\left(n+1\right)}{2}\notin Z^{+}$, a contradiction.

**Theorem 2.3**: G [$C\_{3}$] is (0, 1)-distance magic graph if G is a complete graph.

**Proof**: Since each vertex in $C\_{3}$ replaces every vertex $x\_{i}$ of G and $C\_{3}$ is complete , G is complete .

That implies G[$C\_{3}$] is a complete graph. We know that complete graphs are (0, 1)-distance graph [10]

**Theorem 2.4**$ :P\_{n}[C\_{3}]$ is not (1)-distance magic graph

**Proof**: In any $i^{th}$ copy let a , b are the vertices. Then $N\_{1}(a)$ has all the vertices at a distance 1 but not a and $N\_{1}(b)$ has all the vertices at a distance 1 but not b But then w(a) is not equal to w(b) That implies $P\_{n}[C\_{3}]$ is not (0, 1)-distance graph.

**Theorem 2.5** $: K\_{n\_{1,}n\_{2,}n\_{3},n\_{4}}$ is a (0, 2) –distance magic graph if i) $n\left(n+1\right)≡0 (mod$ 8) where $n=n\_{1}+n\_{2}+n\_{3}+n\_{4}$

 ii)$ n\_{2}=n\_{3}=n\_{4}=s=even, n\_{1}=s-1$

**Proof**: Since $n\left(n+1\right)≡0 (mod$ 8) 8k=n(n+1)

K=$\frac{n(n+1)}{2}$

Assign numbers to the vertices of partite sets as follows.

|  |  |  |  |
| --- | --- | --- | --- |
| Ist partite set | 2nd partite set | 3rd partite set | 4th partite set |
| n | n-1 | n-2 | n-3 |
| n-7 | n-6 | n-5 | n-4 |
| n-8 | n-9 | n-10 | n-11 |
| ... | ... | ... | n-12.. |
| - | 1 | 2 | 3 |

Table 1 (labeling of vertices of $ K\_{n\_{1,}n\_{2,}n\_{3},n\_{4}}$)

**Theorem 2.6** $: C\_{2n}^{n-1}$ is (0, 2) –distance magic graph.

**Proof**: We know that every cycle $C\_{2n}$ is (0, n)-graph.

That implies for any $x\in C\_{2n}$ there exists $y\in C\_{n}$ such that d(x, y)= n

 And $N\_{\left(0,n\right)}\left(x\right)=N\_{\left(0,n\right)}\left(y\right)=\left\{x,y\right\}$

That implies w(x)= w(y)

Now,$ C\_{2n}^{n-1}$is a graph where all the vertices at a distance (n-1) are adjacent

The vertices which are at a distance n in $C\_{2n}$ is at a distance 2 in $ C\_{2n}^{n-1}$

With the same labeling of $C\_{2n} $ we see that $ C\_{2n}^{n-1}$ is (0, 2) – graph.

**Theorem 2.7**$: C\_{4}+\overbar{K\_{m}}$ is (0, 2) –distance magic graph iff $n\left(n+1\right)≡0(mod 6)$ where n=m+4, m=1, 2, 4, 5.

**Proof**: Let a, b, c, d are the 4 vertices of $C\_{4}$ and d(a, c)=2,d(b, d)=2.

Let $v\_{1},v\_{2},…,v\_{n}$ are the vertices of $\overbar{K\_{m}}$.

Note that $N\_{\left(0,2\right)}\left(a\right)= N\_{\left(0,2\right)}\left(c\right)=\{a,c\}$

$$N\_{\left(0,2\right)}\left(b\right)= N\_{\left(0,2\right)}\left(d\right)=\{b,d\}$$

And $N\_{\left(0,2\right)}\left(v\_{i}\right)=V,V=\left\{v\right\}for any i=1,2,…,m$

Suppose given graph is (0, 2)-graph then

w(a)=w(c)=f(a)+f(c)=k

w(b)=w(d)=f(b)+f(d)=k

$$w\left(v\_{i}\right)=\sum\_{}^{}f\left(v\_{i}\right)=k, i=1,2,…,m$$

⇒3k=$\frac{n(n+1)}{2}$

⇒k=$\frac{n(n+1)}{6}$

If $n(n+1)≇0(mod6)$

Then 6∤n(n+1)

⇒k∉$Z^{+}$, a contradiction.

Further for m=3, 6, 9 ,

 $n(n+1)≇0(mod6)$

If m=7, 8, 10, 11, 12,…

$n+(n-1)<\frac{n(n+1)}{6}$=k

Therefore it is not possible to label the vertices a, c or b, d such that

$$f\left(a\right)+f\left(c\right)=\frac{n(n+1)}{6}=k$$

Or f(b)+f(d)=k

Conversely,

 If m=1

$$C\_{4}+\overbar{K\_{1}}=W\_{5}$$

(0, 2) labeling of $W\_{5}$ is given below

1

3

2

4

5

 Figure:2

When m=2 ,n=6 assign f(a)=6, f(c)=1

f(b)=5, f(d)=2

$$f(v\_{1})=3, f\left(v\_{2}\right)=4$$

For m=4, n=8

Assign f(a)=8, f(c)=4

f(b)=7, f(d)=5

$$f\left(v\_{1}\right)=1, f\left(v\_{2}\right)=2, f\left(v\_{3}\right)=3, f\left(v\_{4}\right)=6$$

For m=5, n=9

Assign f(a)=9, f(c)=6

f(b)=8, f(d)=7

$$f\left(v\_{1}\right)=1,f\left(v\_{2}\right)=2,f\left(v\_{3}\right)=3,$$

$$f\left(v\_{4}\right)=4,f\left(v\_{5}\right)=5$$

**Theorem2.8**$: C\_{6}+\overbar{K\_{m}}$ is (0, 2) –distance magic graph iff $n\left(n+1\right)≡0(mod 6)$ where n=m+6, m=2, 3, 5, 6.

**Proof**: Let a, b, c, d, e, f are the vertices of $C\_{6},$ where a and b are adjacent vertices. e and d are at a distance 2 from a and c &f are at a distance 2 from b.

$v\_{i}, 1\leq i\leq m $ are vertices of $K\_{m}$.

Observe that $N\_{\left(0, 2\right)}\left(a\right)=${a, e, d}=$N\_{\left(0,2\right)}\left(e\right)=N\_{\left(0,2\right)}\left(d\right)$

$N\_{\left(0,2\right)}\left(b\right)=${c,f,b}=$N\_{\left(0,2\right)}\left(c\right)=N\_{\left(0,2\right)}\left(f\right)$

$$N\_{\left(0,2\right)}\left(v\_{i}\right)=V, V=\left\{v\_{i}\right\}, i=1,2,…,m$$

Suppose given graph is (0, 2)-graph then

w(a)=w(e)=w(d)=f(a)+f(e)+f(d)=k

w(b)=w(c)=w(f)=f(b)+f(c)+f(f)=k

$$w\left(v\_{i}\right)=\sum\_{}^{}f\left(v\_{i}\right)=k, i=1,2,…,m$$

⇒3k=$\frac{n(n+1)}{2}$

⇒k=$\frac{n(n+1)}{6}$

Now, if $n(n+1)≇0(mod6)$

Then 6∤n(n+1)

⇒k∉$Z^{+}$, a contradiction.

For m=1,4,7,8,9,…. either 6∤n(n+1) or 1+2+3+4…..+m < k

Conversely,

When m=2,

Assign f(a)= 1, f(e)=3, f(d)=8

f(c)=2, f(f)=4, f(b)=6

$$f(v\_{1})=5 , f\left(v\_{2}\right)=7$$

When m=3

Assign f(a)=1 , f(e)=5, f(d)=9

f(c)=2, f(f)=7, f(b)=6

$$f\left(v\_{1}\right)=3, f\left(v\_{2}\right)=4, f\left(v\_{3}\right)=8$$

When m=5

Assign f(a)=9, f(e)=6, f(d)=7

f(c)=11, f(f)=10, f(b)=1

$$f\left(v\_{1}\right)=2, f\left(v\_{2}\right)=3, f\left(v\_{3}\right)=4, f\left(v\_{4}\right)=5, f\left(v\_{5}\right)=8$$

When m=6

Assign f(a)=10 ,f(e) =9, f(d)=7

f(c)=12, f (f)=8, f(b)=6

$$f\left(v\_{1}\right)=1, f\left(v\_{2}\right)=2, f\left(v\_{3}\right)=3, f\left(v\_{4}\right)=4, f\left(v\_{5}\right)=5, f\left(v\_{6}\right)=11$$

III. Conclusion

In this paper we have studied D\_DML of graphs obtained by some graph operations such as join of two graphs, composition etc. It is interesting to check D-DML of graphs obtained by some other graph operations.

References

[1] A.O’Neal, P.Slater, Uniqueness of vertex magic constants, Siam J. Discrete

 Math.27(2013)708-716.

[2] D.B. West, Introduction to Graph Theory, Second Edition, PHI Learning Private Limited(2012).

[3] F. Harary, Graph theory, Narosa Publishing House, 1988.

[4] J.A. Gallian, A dynamic survey of graph labeling, Elec., J. Combin., DS6.(2018).

[5] MirkaMiller, ChrishRodger, Rinovia Simanjuntak, Distance magic labelings of graphs, volume 28(2003).

[6] Muhammad Kashif Shafiq,Gohar Ali,Distance magic labeling of a union of Graphs, 2009

[7] Rachel Rupnow, A survey of distance magic graphs , 2014.

[8] Rinovia Simanjuntak, Mona Elviyenti, Magic labeling of distance atmost2, 2012.

[9] S.Armugam, D.Froncek and N.Kamatchi, Distance magic graphs-a survey, J. Indones. Math. Soc., Special Edition(2011):11-26.

[10]Shankaran P, Chaithra K, D-distance magic labeling on some class of graphs, IJET, 2018 pp(865-866).

[11]Tarkeshwar Singh, d-Distance Neighborhood Magic Graphs.